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The Application of Toroidal Functions for Computing the Magnetic Field from Axially-Polarized Cylindrical Permanent Magnets

Introduction

The off-axis magnetic field from an axially-polarized cylindrical permanent magnet is usually formulated in terms of elliptic integrals. However, an alternate solution, which employs a toroidal harmonic expansion is developed, which is numerically as well as analytically attractive. First, the toroidal expansion converges rapidly and each term in the series is a well-behaved elementary function. Second, because of the rapid convergence of the series, it is most useful for performing a parametric study where one may be interested in optimizing a design. Not many analytical solutions exist for computing the off-axis magnetic induction field from circular cylindrical permanent magnets [1]. Most employ a combination of numerical and analytical techniques [2] to [3]. The method developed in this paper has been known for years [4] to [9], however, not until recently has it gained attention both the physics and engineering communities [10] to [17].

Axially-polarized circular cylindrical permanent magnets are a very common and easily manufactured design which have a wide range of industrial applications. Developing accurate analytical tools to handle off-axis field computations can only help to expedite the design and optimization process. This, invariably, results in cost savings during the manufacturing process.

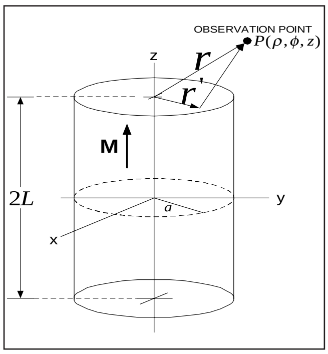

Figure 1. Axially Magnetized Permanent Magnet

Basic Theory and Formulation

Figure 1 represents a simplified model of an axially-polarized circular cylindrical permanent magnet.

The magnetization vector, ***M***, is assumed to be uniform and axially directed as shown in Figure 1. However, this need not be the case. The magnetic field from a permanent magnet can be solved by the introduction of a magnetic scalar potential function and the magnetic charge density model [18]. The magnetic scalar potential function, valid for some external point P from a circular cylindrical permanent magnet as shown in Figure 1 can be written as

$$\Phi_P(\rho,\phi,z) = -\frac{1}{4\pi}\iiint_{V'}\frac{\rho_M}{|\mathrm{r}-\mathrm{r}'|}dV' + \frac{1}{4\pi}\iint_{S'}\frac{\sigma_M}{|\mathrm{r}-\mathrm{r}'|}dS' \qquad (1)$$

where the normal vectors on the top and the bottom cylindrical caps are $\pm\hat{z}$ respectively. Also, the normal unit vector on the cylindrical shell is $\hat{n}' = \hat{\rho}'$. The charge densities are computed as follows:

$$\rho_M = \nabla' \cdot \boldsymbol{M} \ , \quad \text{and} \qquad (2)$$

$$\sigma_M = \hat{\boldsymbol{n}}' \cdot \boldsymbol{M} \ . \qquad (3)$$

The equivalent magnetic volume charge density and the equivalent magnetic surface charge density are given by eq. (2) and eq. (3) respectively. For a uniform magnetization vector, $\boldsymbol{M} = M_o\hat{z}$, the volume charge density, ρ_M, is identically zero and the surface charge density, σ_M, is also zero on the circular shell, since the dot-product between the normal vector on circular shell and the z-directed magnetization vector is zero. However, on the circular cylindrical caps at $z = \pm L$, the magnetic surface charge density is given by

$$\sigma_M = \pm M_o \ , \qquad (4)$$

where the positive and negative sign in eq. (4) refer to the top and bottom cap respectively. The distance between the source point, (ρ,ϕ,z), located on the surface of the top cap and the field point, (ρ',ϕ',L), expressed in cylindrical coordinates, is given by

$$|\boldsymbol{r}-\boldsymbol{r}'| = \sqrt{\rho^2 + \rho'^2 + (z-L)^2 - 2\rho\rho'\cos(\phi-\phi')} \ . \qquad (5)$$

Also, the distance between the source point (ρ,ϕ,z), located on the surface of the bottom cap and the field point, $(\rho',\phi',-L)$, expressed in cylindrical coordinates, is given by

$$|\boldsymbol{r}-\boldsymbol{r}'| = \sqrt{\rho^2 + \rho'^2 + (z+L)^2 - 2\rho\rho'\cos(\phi-\phi')} \ . \qquad (6)$$

One can now employ the known toroidal harmonic expansion for $|\boldsymbol{r}-\boldsymbol{r}'|^{-1}$ [5] to [6], [11], [13] in order to simplify eq. (1).

$$\Phi_P = \frac{M_o}{4\pi^2\sqrt{\rho}}\sum_{m=0}^{\infty}\varepsilon_m\int_0^a\int_0^{2\pi}\left[Q_{m-\frac{1}{2}}(\xi_1) - Q_{m-\frac{1}{2}}(\xi_2)\right]\sqrt{\rho'}\cos\left[m(\phi-\phi')\right]d\phi' d\rho' \ , \qquad (7)$$

where $|\boldsymbol{r}-\boldsymbol{r}'|^{-1}$ is written in terms of a toroidal expansion given by

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{1}{\pi\sqrt{\rho\rho'}}\sum_{m=0}^{\infty}\varepsilon_m Q_{m-\frac{1}{2}}(\xi)\cos\left[m(\phi-\phi')\right] \ . \qquad (8)$$

The $Q_{m-\frac{1}{2}}(\xi)$ is called a Legendre function of the second kind and of half-integral degree [19] or a toroidal function of zeroth order [20]. They are also referred to as Q-functions [6]. The argument of the toroidal function for the top cap is $\xi_1 = \frac{\rho^2 + \rho'^2 + (z-L)^2}{2\rho\rho'} > 1$, and for the bottom cap is $\xi_2 = \frac{\rho^2 + \rho'^2 + (z+L)^2}{2\rho\rho'} > 1$. The Neumann factor [21], ε_m, is 1 for m=0 and 2 for all m≥1. The Q-function

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can be expanded as follows:

$$Q_{m-\frac{1}{2}}(\xi) = \frac{\pi}{(2\zeta)^{m+\frac{1}{2}} 2^m} \sum_{n=0}^{\infty} \frac{(4n+2m-1)!!}{2^{2n}(n+m)!n!} \frac{1}{(2\zeta)^{2n}}, \tag{9}$$

where (4n+2m-1)!!=1·3·5·7···(4n+2m-1) for all m, n ≥ 0. In eq. (7), only the m=0 term survives the ϕ' integration. This leads to the following magnetic scalar potential function:

$$\Phi_P = \frac{M_o}{2\pi\sqrt{\rho}} \int_0^a [Q_{-\frac{1}{2}}(\xi_1) - Q_{-\frac{1}{2}}(\xi_2)]\sqrt{\rho'}\, d\rho', \tag{10}$$

where

$$Q_{-\frac{1}{2}}(\xi_i) = \pi \sum_{n=0}^{\infty} \frac{(4n-1)!!}{2^{2n}(n!)^2} \left(\frac{\rho\rho'}{\rho^2 + \rho'^2 + (z-z_i)^2} \right)^{2n+\frac{1}{2}} \tag{11}$$

and $i = 1$ for the top cap and $i = 2$ for the bottom cap. Employing eq. 11 and eq. 10, the magnetic scalar potential becomes

$$\Phi_P = \frac{M_o}{2} \sum_{n=0}^{\infty} \frac{(4n-1)!!\rho^{2n}}{2^{2n}(n!)^2} \int_0^a \left\{ \frac{(\rho')^{2n+1}}{(\rho^2+\rho'^2+(z-L)^2)^{2n+\frac{1}{2}}} - \frac{(\rho')^{2n+1}}{(\rho^2+\rho'^2+(z+L)^2)^{2n+\frac{1}{2}}} \right\} d\rho'. \tag{12}$$

The interchange of summation and integration is valid since the series in eq. (12) is uniformly convergent for $\xi_i > 1$.

The integral in eq. (12) can be expressed in terms of hypergeometric functions [22]. This leads to the final form for the magnetic scalar potential.

$$\Phi_P = \frac{M_o a^2}{4} \sum_{n=0}^{\infty} \frac{(4n-1)!!(a\rho)^{2n}}{2^{2n}(n+1)(n!)^2} \left\{ \frac{\mathrm{F}\left(n+1, 2n+\frac{1}{2}; n+2; -\frac{a^2}{\rho^2+(z-L)^2}\right)}{(\rho^2+(z-L)^2)^{2n+\frac{1}{2}}} - \frac{\mathrm{F}\left(n+1, 2n+\frac{1}{2}; n+2; -\frac{a^2}{\rho^2+(z+L)^2}\right)}{(\rho^2+(z+L)^2)^{2n+\frac{1}{2}}} \right\} \tag{13}$$

Equation (13) represents the external magnetic scalar potential from a finite circular cylindrical axial-polarized permanent magnet. The hypergeometric functions, **F**(•), in eq. (13), can be evaluated for any value of n. For example, when n=0 in eq. (13) one obtains

$$\Phi_P^{(n=0)} = \frac{M_o}{2} \left\{ \left(\sqrt{\rho^2+a^2+(z-L)^2} - \sqrt{\rho^2+(z-L)^2} \right) - \left(\sqrt{\rho^2+a^2+(z+L)^2} - \sqrt{\rho^2+(z+L)^2} \right) \right\}. \tag{14}$$

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One can do a simple check to see that the magnetic induction field at points external to the magnet and which lie on the z-axis are given by

$$B_{\rho}^{(\rho \to 0)} = 0, \tag{15}$$

$$B_{z}^{(\rho \to 0)} = \frac{\mu_0 M_o}{2}\left\{\frac{(z+L)}{\sqrt{a^2+(z+L)^2}} - \frac{(z-L)}{\sqrt{a^2+(z+L)^2}}\right\}. \tag{16}$$

One can obtain closed-form expressions, in terms of elementary functions, for the external magnetic scalar potential function, $\Phi_P^{(n)}$ for any value of n in eq. (13). With symbolic mathematical tools such as Mathematica [23] or Maple [24], one can easily find as many terms in the expansion of eq. (13) as desired. One can rewrite the total magnetic scalar potential as

$$\Phi_P^{(N)} = \sum_{k=0}^{N} \Phi_P^{(n=k)}, \tag{17}$$

where N is large enough to approximate eq. (13). Since the Q-functions converge rapidly [25], eq. (13) can accurately be approximated by eq. (17) for relatively small values of N even at observation points which are close to the magnet. The example problem in the next section illustrates the process.

Once the magnetic scalar potential is known, the external magnetic flux density can be computed from

$$\boldsymbol{B} = -\mu_0 \nabla \Phi_{\boldsymbol{P}} = -\mu_0\left(\hat{\rho}\frac{\partial}{\partial \rho} + \hat{\phi}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z}\right)\Phi_P \tag{18}$$

Taking the gradient of eq. (13) in cylindrical coordinates results in somewhat lengthy expressions for the components of magnetic induction field as n increases. However, these expressions are easily computed and plotted using Mathematica or Maple.

Illustrative Example

Consider, for example, a solid circular cylindrical axially-polarized permanent magnet with a radius of 0.5 of an inch, a length of 1 inch, a magnetization of 10^5 A/m, and modeled as shown in Figure 1. All dimensions were converted to the MKS system of units before computations were carried out. In order to compute the components of the magnetic flux density in cylindrical coordinates at any arbitrary point in space external to the permanent magnet, eq. (17) and eq. (18) are employed.

Consider enclosing the permanent magnet in a hypothetical observation cylinder. This cylinder is simply a grid of observation points where the magnetic scalar potential and the three components of the magnetic flux density are computed. In this example, a total of 4,453 points were used. Each cap has 1,460 observation points and the cylindrical shell has 1,533 observation points. The maximum radius of the observation cylinder is 0.55 of an inch and it has a length of 1.1 inches. The permanent magnet is centered at the origin as shown in Figure 1 and the observation cylinder completely encloses the magnet. All plots were developed using using Tecplot [26].

Consider, for example, the choice of N=25 in eq. (17). The scalar potential and the field components, for this value of N, are shown in Figures 2 through 4.

One can easily check to see whether, for example, the magnetic scalar potential for N=25 is accurate. Figures 5 and 6 represent the radial and axial induction fields for N=40 in eq. (17).

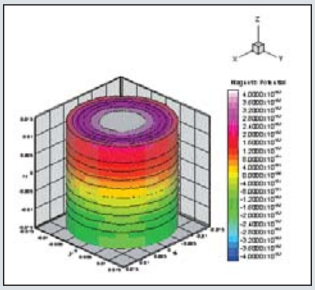

Figure 2. Magnetic Scalar Potential $\Phi_P^{(N=25)}$

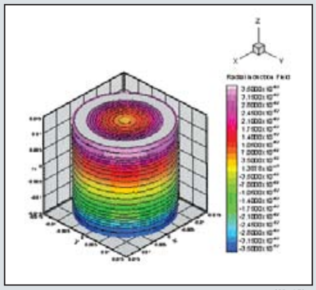

Figure 3. Radial Induction Field $B_\rho^{(N=25)}$

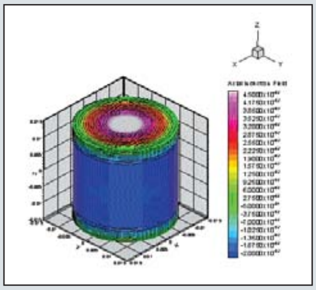

Figure 4. Axial Induction Field $B_z^{(N=25)}$

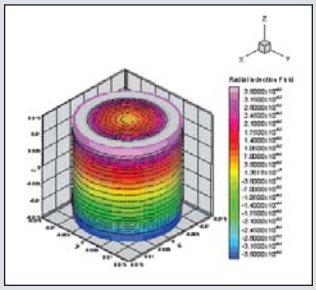

Figure 5. Radial Induction Field $B_\rho^{(N=40)}$

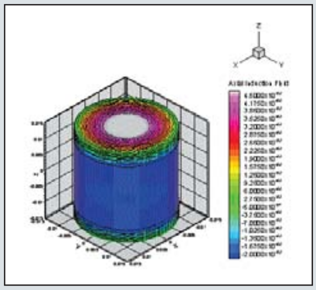

Figure 6. Axial Induction Field $B_z^{(N=40)}$

One can see that Figures 3 and 5 and Figures 4 and 6 are virtually identical. One may not need to compute many terms in the series given in eq. (17). This is an important feature when a parametric study is undertaken because one wants to be able to make rapid changes in various parameters without having to wait a long time for the computations to complete. This is one reason why the finite element method and others are not well suited for parametric studies except for relatively simple problems. Also, the authors chose a large number of observation points and each point has an associated computation time. One could easily have chosen fewer observation points without loss of accuracy and this would lessen the computation time.

Discussion and Conclusion

A method has been proposed for computing the three-dimensional magnetic field from an axially-polarized circular cylindrical permanent magnet. This method relies on the use of toroidal harmonics. Although the final result was in the form of an infinite series, the series converges quite rapidly even for observation points relatively close to the magnetic source. An example is given that illustrates the three-dimensional field pattern exhibited by each component of the magnetic induction field for N=25. Because the infinite series solution converges rapidly, only a few terms are needed to produce an accurate field pattern. This was clearly shown by computing the components of the induction field for N=40.

The main purpose of this article is to develop a method of solution for rapidly computing the magnetic induction field produced by an axially-polarized cylindrical permanent magnet. The use of the toroidal harmonic solution allows one to optimize the design process without sacrificing speed. One can rapidly change the various design parameters, such as magnetization, physical geometry, observation cylinder, etc., quickly enough to modify a design. The implementation of powerful symbolic mathematical tools such as Maple and Mathematica have played an integral role in the application of the toroidal harmonic expansion.

There are a few methods [2] to [3] for performing optimization for cylindrical magnets but most rely on a combination of numerical as well as analytical methods. However, a parametric study is most easily accomplished with an analytical solution. In most real-world engineering problems, a completely analytical solution most often does not exist. However, the method introduced in this paper should provide a useful analytical tool for accurately predicting the magnetic field from circular cylindrical permanent magnets.

Dr. Selvaggi received Bachelor's degrees in electrical and mechanical engineering from Pratt Institute in Brooklyn New York. He has masters degrees in electrical power engineering and mechanical engineering from Rensselaer Polytechnic Institute. He has a masters degree from Courant Institute of mathematical sciences at New York University. Also, he has a PhD in electrical power engineering from Rensselaer Polytechnic Institute specializing in electromagnetics. He can be reached at jpsst44@bellatlantic.net.

REFERENCES

[1] J. P. Selvaggi, "Computation of the Three-Dimensional Magnetic Field From Solid Permanent-Magnet Bipolar Cylinders By Employing Toroidal Harmonics," IEEE Transactions on Magnetics, vol. 43, No. 10, pp. 3833-3839, October 2007.

[2] E.P. Furlani and M.A. Knewtson, "A three-dimensional field solution for permanent-magnet axial-field motors," IEEE Transactions on Magnetics, vol. 33, Issue 3, pp. 2322-2325, May 1997.

[3] E.P. Furlani, Permanent Magnet and Electromechanical Devices, Academic Press, 2001.

[4] E. Heine, Handbuch der Kugelfunctionen-Theorie und Anwendungen, Druck und Verlag von G. Reimer, 1881.

[5] E.W. Hobson, "On Green's Function for a Circular Disk, with applications to Electrostatic Problems," Transactions of the Cambridge Philosophical Society, vol.18, pp. 277--291, 1900.

[6] C.J. Bouwkamp and N.G. de Bruijn, "The Electrical Field of a Point Charge Inside a Cylinder, in Connection with Wave Guide Theory," Journal of Applied Physics, vol. 18, pp. 562--577, June 1947.

[7] C. Snow, "Magnetic Fields of Cylindrical Coils and Annular Coils," Natl. Bur. Stand., Applied Mathematics Series 38, pp. 1--29, 1953.

[8] C. Snow, Formulas for Computing Capacitance and Inductance, National Bureau of Standards Circular 544, 1954.

[9] C. Snow, Hypergeometric and Legendre Functions with Applications to Integral Equations of Potential Theory, National Bureau of Standards Applied Mathematics Series 19, 1952.

[10] H.S. Cohl, J.E. Tohline, and A.R.P. Rau, "Developments in determining the gravitational potential using toroidal functions," Astron. Nachr. vol. 321, pp. 363--372, 2000.

[11] H.S. Cohl and J.E. Tohline, "A Compact Cylindrical Green's Function Expansion For The Solution Of Potential Problems," The Astrophysical Journal vol. 527, pp. 86--101, Dec. 1999.

[12] H.S. Cohl, A.R.P. Rau, J.E. Tohline, D.A. Browne, J.E. Cazes, and E.I. Barnes, "Useful alternative to the multipole expansion of 1/r potentials," Physical Review A, vol. 64, pp. 052509-1--5, 2001.

[13] J. Selvaggi, S. Salon, O. Kwon, and M.V.K. Chari, "Calculating the External Magnetic Field From Permanent Magnets in Permanent-Magnet Motors-An Alternative Method," IEEE Transactions on Magnetics, vol. 40, no. 5, pp. 3278--3285, Sep. 2004.

[14] J. Selvaggi, "Multipole Analysis of Circular Cylindrical Magnetic Systems," Ph.D. dissertation, Rensselaer Polytechnic Institute, Troy NY, 2005.

[15] J. Selvaggi, S. Salon, O. Kwon, and M.V.K. Chari, "Application of a Toroidal Harmonic Expansion for Computing the Magnetic Field from a Balanced 6-Pole Permanent-Magnet Motor," ICEM 2006, XVII International Conference on Electrical Machines , Paper Code PSA1-20, 2006.

[16] J. Selvaggi, S. Salon, O. Kwon, and M.V.K. Chari, "A General Method for Calculating the External Magnetic Field from a Cylindrical Magnetic Source using Toroidal Functions," Electric Machines Technology Symposium, Renaissance Hotel, Philadelphia, PA. May 22-24, 2006.

[17] J. Selvaggi, S. Salon, and M.V.K. Chari, "An application of toroidal functions in electrostatics," Am. J. Phys., vol. 75, no. 8, pp. 724-727, August 2007.

[18] J.D. Jackson, Classical Electrodynamics- 3^{rd} edition, John Wiley and Sons, 1999.

[19] N.N. Lebedev, Special Functions and Their Applications, Prentice-Hall, 1965.

[20] M. Abramowitz, and I. Stegun, Handbook of Mathematical Functions, Dover Publications Inc., 1972.

[21] P. M. Morse, H.Feshbach, Methods of Theoretical Physics I &II, McGraw Hill, 1953.

[22] E.T. Whittaker, G.N Watson, A Course of Modern Analysis-4^{th} edition, Cambridge University Press, 1952.

[23] Wolfram Research Inc., Mathematica, Version 5.0, Champaign, IL, 100 Trade Center Drive, 61820-7237, USA, 2003.

[24] Waterloo Maple Inc., Maple, Version 10.06, Waterloo, Ontario, Canada N2L 6C2, 2004.

[25] H.E. Fettis, "A New Method for Computing Toroidal Harmonics," Mathematics of Computation 24, No. 111, July 1970.

[26] Tecplot Inc., Tecplot 360, Ver.11.0, Bellevue, WA, 3535 Factoria Blvd., S.E. Suite 550 98006, 2006.

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